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Lacunary Convergence of Sequences of Complex Uncertain Variables

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ABSTRACT

This paper introduces the notion of strongly Cesàro summable sequences, the sequences of uniformly strongly Cesàro summable and the strong almost convergent sequences of complex uncertain sequences. A study on the lacunary strong convergence concepts of sequences of complex uncertain variables of different types have also been done.

Keywords: Almost convergence, Cesàro summable sequences, complex uncertain variable, uncertainty theory.

1. Introduction

The notion of uncertainty was introduced by B. Liu in the year 2007 (one may refer to Liu (2016) or Liu (2010)) which is based on an uncertain measure which satisfies normality, duality, subadditivity and product axioms. It has been applied to different areas, such as uncertain programming, uncertain risk analysis, uncertain differential equation, uncertain finance, uncertain optimal control, uncertain game, uncertain graph etc. For more details of uncertain theory one may refer to Liu (2016). The concept of uncertain variable defined on the uncertain space was developed in order to define uncertain sequences. Complex uncertain sequences are measurable functions from an uncertain space to the set of complex numbers. Sequence convergence, as a fundamental theory in mathematics, has drawn the attention of many researcher towards the study of convergence concepts of uncertain sequences.You (2009) first introduced the concept of convergence in mean, convergence in measure and convergence in distribution and derived the relations between them. Some works on sequences of uncertain variables are also done by Chen et al. (2016), You and Yan (2017), Tripathy and Nath (2017) and Tripathy and Dowari (2018).

The initial work on lacunary sequence is found in Freedman et al. (1978). They studied strongly Cesàro summable sequences with general lacunary θ results in a larger class N_θ in the $BK-$ spaces(a Banach space of sequences for which the co-ordinate linear functionals are continuous). Further lacunary sequences have been investigated by Tripathy and Mahanta (2004), Tripathy and Baruah (2010) and Tripathy et al. (2012). Our main interest here would be to study the classes with the uncertain sequences in the uncertainty space and the lacunary convergence concepts of complex uncertain sequences.

2. Preliminaries

Let L be a $\sigma-$algebra on a nonempty set Γ. A set function M is called an uncertain measure if it satisfies the following axioms:
Axiom 1 (Normality Axiom). $M\{\Gamma\} = 1$.
Axiom 2 (Duality Axiom). $M\{\Lambda\} + M\{\Lambda^c\} = 1$ for any $\Lambda \in L$.
Axiom 3 (Subadditivity Axiom). For every countable sequence of $\{\lambda_j\} \in L$, we have

$$M\left\{\bigcup_{j=1}^{\infty} \lambda_j\right\} \leq \sum_{j=1}^{\infty} M\{\lambda_j\}.$$

The triplet (Γ, L, M) is called an uncertainty space, and each element Λ in L is called an event. In order to obtain an uncertain measure of compound event, a product uncertain measure is defined by Liu (2016) as follows:

Axiom 4 (Product Axiom). Let (Γ_k, L_k, M_k) be uncertainty space for $k = 1, 2, 3, ...$ The product uncertain measure M is an measure satisfying

$$M\left\{\prod_{k=1}^{\infty} \Lambda_k\right\} = \bigwedge_{k=1}^{\infty} M_k\{\Lambda_k\}$$

where Λ_k are arbitrarily chosen events from L_k for $k = 1, 2, ...$, respectively.

A complex uncertain variable is a measurable function ξ from an uncertainty space (Γ, L, M) to the set of complex numbers, i.e., for any Borel set B of complex numbers, the set

$$\{\xi \in B\} = \{\gamma \in \Gamma : \xi(\gamma) \in B\}$$

is an event. When the range is the set of real numbers, we call it as an uncertain variable, introduced and investigated by Liu (2016). As a complex function on uncertainty space, complex uncertain variable is mainly used to model a complex uncertain quantity.

The expected value operator of an uncertain variable was defined by Liu (2016) as

$$E[\xi] = \int_0^{+\infty} M\{\xi \geq r\}dr - \int_{-\infty}^{0} M\{\xi \leq r\}dr$$

provided that at least one of the two integrals is finite.

The complex uncertainty distribution $\Phi(x)$ of a complex uncertain variable ξ is a function from $\mathbb{C}$ to $[0, 1]$ defined by

$$\Phi(c) = M\{Re(\xi) \leq Re(c), Im(\xi) \leq Im(c)\}$$

for any complex number c.

An uncertain variable is said to be positive, when it maps from $\mathbb{R}_+ \bigcup\{0\}$(non-negative real numbers) to $[0, 1]$. Considering the important role of sequence convergence in mathematics, some concepts of convergence for complex uncertain sequences were introduced by Chen et al. (2016). Complex uncertain sequences are sequence of complex uncertain variables indexed by integers.

The complex uncertain sequence $\{\xi_n\}$ is said to be *convergent almost surely (a.s.) to* ξ if there exists an event Λ with $M\{\Lambda\} = 1$ such that

$$\lim_{n\to\infty} \|\xi_n(\gamma) - \xi(\gamma)\| = 0,$$

for every $\gamma \in \Lambda$. In that case we write $\xi_n \to \xi$, a.s.

The complex uncertain sequence $\{\xi_n\}$ is said to be *convergent in measure to* ξ if for a given $\varepsilon > 0$,

$$\lim_{n\to\infty} M\{\|\xi_n - \xi\| \geq \varepsilon\} = 0.$$

The complex uncertain sequence $\{\xi_n\}$ is said to be *convergent in mean to* ξ if

$$\lim_{n\to\infty} E[\|\xi_n - \xi\|] = 0.$$

Let $\Phi_1, \Phi_2, \Phi_3, ...$ be the complex uncertainty distributions of complex uncertain variables $\xi_1, \xi_2, \xi_3...$, respectively. We say the complex uncertain sequence $\{\xi_n\}$ *converges in distribution to* ξ if

$$\lim_{n\to\infty} \Phi_n(c) = \Phi(c)$$

for all $c \in \mathbb{C}$, at which $\Phi(c)$ is continuous.

The complex uncertain sequence $\{\xi_n\}$ is said to be *convergent uniformly almost surely (u.a.s.) to* ξ if there exists a sequence of events $\{E'_k\}$, $M\{E'_k\} \to 0$ such that $\{\xi_n\}$ converges uniformly to ξ in $\Gamma - E'_k$, for any fixed $k \in \mathbb{N}$.

3. Definition and Results

By a lacunary sequence $\theta = (k_r)$; where $k_0 = 0$ we shall mean an increasing sequence of non-negative integers with $k_r - k_{r-1} \to \infty$, as $n \to \infty$. The intervals determined by θ will be denoted by $I_r = (k_{r-1}, k_r]$ and $q_r = \frac{k_r}{k_{r-1}}$, $h_r = k_r - k_{r-1}$ for $r = 1, 2, 3, ...$
Sums of the form $\sum\limits_{i=k_{r-1}+1}^{k_r} |x_i| = \sum\limits_{i\in I_r} |x_i|$, will often be written for convenience as $\sum\limits_{I_r} |x_i|$.

Here we define the space $|\sigma_1|$ of strongly Cesàro summable complex uncertain sequence by

$|\sigma_1| = \{\xi = (\xi_i(\gamma))$:there exists $l(\gamma)$ such that $\frac{1}{n}\sum_{i=1}^{n}\|\xi_i(\gamma) - l(\gamma)\| \to 0,$ as $n \to \infty\}$.

Definition 3.1. *Let $\theta = (k_r)$ be a lacunary sequence and (ξ_k) be a sequence of uncertain variables in the space (Γ, L, M),*
$N_\theta = \{\xi = (\xi_k)$ *: there exists $l(\gamma)$ such that* $\tau_r \equiv \frac{1}{h_r}\sum_{k\in I_r}\|\xi_k(\gamma) - l(\gamma)\| \to 0,$ *as* $n \to \infty\}$.

Theorem 3.1. *$|\sigma_1| \subseteq N_\theta$, it is necessary and sufficient that* $\liminf_r \; q_r > 1$.

Proof. For the sufficiency we assume $\liminf_r \; q_r > 1$, then there exists $\delta(\gamma) \in (\Gamma, L, M)$ and $M(\delta(\gamma)) > 0$ such that, $1 + M(\delta(\gamma)) \le q_r$ for all $r \ge 1$.
Now, for $\xi(\gamma) \in |\sigma_1|^0$ we have,

$$\begin{aligned}\tau_r &= \frac{1}{h_r}\sum_{i=1}^{k_r}\|\xi_i(\gamma)\| - \frac{1}{h_r}\sum_{i=1}^{k_{r-1}}\|\xi_i(\gamma)\| \\ &= \frac{k_r}{h_r}(\frac{1}{k_r}\sum_{i=1}^{k_r}\|\xi_i(\gamma)\|) - \frac{k_{r-1}}{h_r}(\frac{1}{k_{r-1}}\sum_{i=1}^{k_r}\|\xi_i(\gamma)\|)\end{aligned}$$

Since $h_r = k_r - k_{r-1}$, we have $\frac{k_r}{h_r} \le \frac{1+M(\delta(\gamma))}{M(\delta(\gamma))}$ and $\frac{k_{r-1}}{h_r} \le \frac{1}{M(\delta(\gamma))}$; as $M(\delta(\gamma)) > 0$ and $q_r = \frac{k_r}{k_{r-1}}$.

The terms $\frac{1}{k_r}\sum_{i=1}^{k_r}\|\xi_i(\gamma)\|$ and $\frac{1}{k_{r-1}}\sum_{i=1}^{k_r}\|\xi_i(\gamma)\|$ both converges to 0.

Hence τ_r converge to 0, that is, $\xi_i(\gamma) \in N_\theta^0$.

Therefore, $|\sigma_1| \subseteq N_\theta$.

For the sufficiency we assume, $\liminf_r q_r = 1$.

Since θ is lacunary, we can find a subsequence k_{r_j} of θ satisfying,
$\frac{k_{r_j}}{k_{r_j-1}} < 1 + \frac{1}{j}$ and $\frac{k_{r_j-1}}{k_{r_{j-1}}} > j$, where $r_j \ge r_{j-1} + 2$.

Define $\xi = (\xi_i(\gamma))$ by

$$\xi_i(\gamma) = \begin{cases} 1, & \text{if } i \in I_{r_j} \text{ for some } j = 1, 2, ...; \\ 0, & \text{otherwise.} \end{cases}$$

Then for any $l(\gamma)$,

$$\frac{1}{h_{r_j}} \sum_{I_{r_j}} \|\xi_i(\gamma) - l(\gamma)\| = \|1 - l(\gamma)\|; j = 1, 2, ...$$

and

$$\frac{1}{h_r} \sum_{I_r} \|\xi_i(\gamma) - l(\gamma)\| = \|l(\gamma)\| \quad \text{for} \quad r \neq r_j.$$

It follows that $(\xi_i(\gamma)) \in N_\theta$.

But, $(\xi_i(\gamma))$ is strongly summable, since if we consider t is sufficiently large,there exists a unique j for which $k_{r_j - 1} < t \leq k_{r_{j+1} - 1}$ and write

$$\frac{1}{t} \sum_{i=1}^{t} \|\xi_i(\gamma)\| \leq \frac{k_{r_{j-1}} + h_{r_j}}{k_{r_j} - 1} \leq \frac{1}{j} + \frac{1}{j} = \frac{2}{j}.$$

Now, if $t \to \infty$, it follows that $j \to \infty$. Hence $(\xi_i(\gamma)) \in |\sigma_1|^0$. □

Theorem 3.2. *$N_\theta \subseteq |\sigma_1|$, it is necessary and sufficient that* $\limsup_r \ q_r < \infty$.

Proof. For the sufficiency we consider $\limsup_r \ q_r$, there exists $H(\gamma) \in (\Gamma, L, M)$ and $M(H(\gamma)) > 0$ such that $q_r < M(H(\gamma))$ for all $r \geq 1$. Considering $(\xi_i(\gamma)) \in N_\theta^0$ and $\varepsilon > 0$ we can find $R > 0$ and $K > 0$ such that $\tau_i < k$ for all $i = 1, 2, ...$

Then if t is any integer with $k_{r-1} < t \leq k_r$, where $r > R$, we can write,

$$\begin{aligned}
\frac{1}{t}\sum_{i=1}^{t}\|\xi_i(\gamma)\| &\le \frac{1}{k_{r-1}}\sum_{i=1}^{t}\|\xi_i(\gamma)\| \\
&= \frac{1}{k_{r-1}}\left(\sum_{I_1}\|\xi_i(\gamma)\| + \cdots + \sum_{I_r}\|\xi_i(\gamma)\|\right) \\
&= \frac{1}{k_{r-1}}\tau_1 + \frac{k_2-k_1}{k_{r-1}}\tau_2 + \cdots + \frac{k_R-k_{R-1}}{k_{r-1}}\tau_r \\
&\quad + \frac{k_{R+}-k_R}{k_{r-1}}\tau_{R+1} + \cdots + \frac{k_r-k_{r-1}}{k_{r-1}}\tau_r \\
&\le (\sup_{i\ge 1}\tau_i)\frac{k_R}{k_{r-1}} + (\sup_{i\ge R}\tau_i)\frac{k_r-k_R}{k_{r-1}} \\
&= k\cdot\frac{k_R}{k_{r-1}} + \varepsilon\cdot M(H(\gamma))
\end{aligned}$$

Since $k_{r-1}\to\infty$ as $t\to\infty$, it follows that
$\frac{1}{t}\sum\limits_{i=1}^{t}\|\xi_i(\gamma)\|\to 0$ and consequently $\xi(\gamma)\in|\sigma_1|^0$.

For the necessity part we consider $\limsup\limits_{r}\ q_r=\infty$ and construct a sequence in N_θ that is not strongly Cesàro Summable.

We select a subsequence (k_{r_j}) of θ so that $q_{r_j}>j$ and then define $\xi=(\xi(\gamma))$ by

$$\xi_i(\gamma)=\begin{cases}1, & \text{if } k_{r_{j-1}}<i\le 2k_{r_j-1}, \text{ for some } j=1,2,...;\\ 0, & \text{otherwise.}\end{cases}$$

Then

$$\tau_{r_j}=\frac{k_{r_j-1}}{k_{r_j}-k_{r_{j-1}}}<\frac{1}{j-1}$$

if $r=r_j$, $\tau_r=0$. Thus $(\xi_i(\gamma))\in N_\theta^0$.

Any sequence in $|\sigma_1|$ consisting of only 0's and 1's has an strong limit $l(\gamma)\in(\Gamma,L,M)$ where the $M(l(\gamma))=1$ or $M(l(\gamma))=0$.

For the sequence $\xi = (\xi_i(\gamma))$ and $i = 1, 2, \cdots, k_{r_j}$.

$$\frac{1}{k_{r_j}} \sum \|\xi(\gamma) - 1\| \geq \frac{1}{k_{r_j}}(k_{r_j} - 2k_{r_j - 1})$$
$$= 1 - \frac{2k_{r_j} - 1}{k_r}$$
$$> 1 - \frac{2}{j}$$

which converge to 1 and for $i = 1, 2, \cdots, 2k_{r_j} - 1$

$$\frac{1}{2k_{r_j} - 1} \sum_i \|\xi(\gamma)\| \geq \frac{k_{r_j - 1}}{2k_{r_j} - 1} = \frac{1}{2}$$

Thus, $(\xi_i(\gamma)) \in |\sigma_1|$. □

3.1 Almost convergent sequences

Let m denote the linear space of bounded sequences. A sequence $x \in m$ is said to be almost convergent and s is called its generalized limit if each Banach Limit of x is s. The class F of almost convergent sequences was introduced by Lorentz (1948), he proved that a sequence $x = \{x_k\}$ is almost convergent if and only if

$$\lim_{p \to \infty} \frac{x_n + x_{n+1} + \cdots + x_{n+p-1}}{p} = s$$

holds uniformly in n.

A convergent sequence is almost convergent with its limit and the generalized limit are identical. For example, the sequence, $(x_n) = (1, 0, 1, 0, ...)$ is not convergent but it is almost convergent to the limit $\frac{1}{2}$.
Lorentz (1948) proved that F is linear, not-separable. F is nowhere dense in m, dense in itself and closed-therefore perfect.
A sequence $x = (x_i)$ is *strong almost convergent* if there exists a number L for which

$$\frac{1}{n} \sum_{i=m+1}^{m+n} |x_i - l| \to 0 \qquad (n \to \infty),$$

uniformly in $m = 0, 1, 2, ...$

Let $|AC|$ denotes the set of strongly almost convergent sequences. The space $|AC|$ contains all convergent sequences and is regular in the sense that the unique l associated with a sequence $x \in |AC|$ agrees with $\lim_n x_n$ in the case where x is convergent. Sequences satisfying the above condition with the absolute value signs removed are the almost convergent sequences as given by King (1966). For more details on almost convergence and certain summabilty methods one may refer to Başar (1992), Başar and Kirişçi (2011), Başar (2012), Yeşilkayagil and Başar (2015).

Here we define the strong almost convergent sequences in terms of uncertain variables in the uncertain space (Γ, L, M).

Definition 3.2. *An uncertain sequence $\xi = (\xi_i)$ in the space (Γ, L, M) is said to be strongly almost convergent if there exists a uncertain variable $l(\gamma) \in (\Gamma, L, M)$ for which*

$$\frac{1}{n}\sum_{i=m+1}^{m+n} \|\xi_i(\gamma) - l(\gamma)\| \to 0 \qquad (n \to \infty),$$

uniformly in $m = 0, 1, 2, ...$

Theorem 3.3. $|AC| \subset N_\theta$.

Proof. Let $(\xi_i(\gamma)) \in |AC|$ and $\varepsilon > 0$,there exists $N > 0$ and l such that

$$\frac{1}{n}\sum_{i=m+1}^{m+n} \|\xi_i(\gamma) - l(\gamma)\| < \varepsilon \quad \text{for} \quad n > N, \quad m = 0, 1, 2, ...$$

As θ is lacunary we can choose $R > 0$ such that $r \geq R$ implies $h_r > N$ and consequently $\tau_r < \varepsilon$. Thus $(\xi_i(\gamma)) \in N_\theta$. Thus to obtain a sequence in N_θ but not in $|AC|$ define $\xi = (\xi_i(\gamma))$ by

$$\xi_i(\gamma) = \begin{cases} 1, & \text{if for some } r, \quad k_{r-1} < i \leq k_{r-1} + \sqrt{h_r}; \\ 0, & \text{otherwise.} \end{cases}$$

Therefore, ξ contains arbitrarily long strings of 0's and 1's, from which it follows that ξ is not strongly almost convergent.
But, $\tau_r = \frac{1}{h_r}\sum_{I_r}\|\xi_i\| = \frac{1}{h_r}[\sqrt{h_r}] = \frac{1}{\sqrt{h_r}}$ which converges to 0 as $r \to \infty$. □

4. Lacunary Convergence of Complex Uncertain Sequences

In this section we define the lacunary convergence concepts of uncertain sequences and derive the relations between them:

Definition 4.1. *The complex uncertain sequence* $\{\xi_k\}$ *is said to be lacunary strongly convergent almost surely to* ξ *if for every* $\varepsilon > 0$ *there exists an event* Λ *with* $M\{\Lambda\} = 1$ *such that*

$$\lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} \|\xi_k(\gamma) - \xi(\gamma)\| = 0,$$

for every $\gamma \in \Lambda$.

Definition 4.2. *The complex uncertain sequence* $\{\xi_k\}$ *is said to be lacunary strongly convergent in measure to* ξ *if*

$$\lim_{r\to\infty} M(\gamma \in \Gamma : \{\frac{1}{h_r} \sum_{k\in I_r} \|\xi_k(\gamma) - \xi(\gamma)\|\} > \varepsilon) = 0,$$

for every $\varepsilon > 0$.

Definition 4.3. *The complex uncertain sequence* $\{\xi_k\}$ *is said to be lacunary strongly convergent in mean to* ξ *if*

$$\lim_{r\to\infty} E[\frac{1}{h_r} \sum_{k\in I_r} \|\xi_k(\gamma) - \xi(\gamma)\|] = 0,$$

for every $\varepsilon > 0$.

Definition 4.4. *Let* $\Phi_1, \Phi_2, \Phi_3, ...$ *be the complex uncertainty distributions of complex uncertain variables* $\xi_1, \xi_2, \xi_3, ...$*, respectively. We say the complex uncertain sequence* $\{\xi_k\}$ *lacunary strong convergent in distribution to* ξ *if for every* $\varepsilon > 0$*,*

$$\lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} \|\Phi_k(c) - \Phi(c)\| = 0,$$

for all complex c *at which* $\Phi(c)$ *is continuous.*

Definition 4.5. *The complex uncertain sequence* $\{\xi_n\}$ *is said to be convergent uniformly almost surely to* ξ *if there exists an sequence of events* $\{E_k\}$*,* $M\{E_k\} \to 0$ *such that* $\{\xi_n\}$ *converges uniformly to* ξ *in* $\Gamma - E_k$*, for any fixed* $k \in \mathbb{N}$.

4.1 Relationships among the Convergence Concepts

Here we discuss the relations among the convergence concepts of complex uncertain sequences.

Lacunary strongly convergent in mean and lacunary strongly convergent in measure

Theorem 4.1. *If the complex uncertain sequence $\{\xi_k\}$ lacunary strongly convergent in mean to ξ, then $\{\xi_k\}$ lacunary strongly converges in measure to ξ.*

Proof. It follows from the Markov inequality that for any given $\varepsilon > 0$, we have
$\lim\limits_{r\to\infty} M(\{\frac{1}{h_r}\sum\limits_{k\in I_r} \|\xi_k(\gamma) - \xi(\gamma)\|\} > \varepsilon)$
$\leq \lim\limits_{r\to\infty} \frac{E[\frac{1}{h_r}\sum_{k\in I_r} \|\xi_k(\gamma)-\xi(\gamma)\|]}{\varepsilon} \to 0$
as $k \to \infty$. Thus $\{\xi_k\}$ lacunary strongly converges in measure to ξ and the theorem is thus proved.

□

But the converse of the above theorem need not be true in general i.e. lacunary strong convergence in measure does not imply lacunary strong convergence in mean. This can be illustrated with the example below.

Consider the uncertainty space (Γ, L, M) to be $\{\gamma_1, \gamma_2, ...\}$ with power set and

$$M\{\Lambda\} = \begin{cases} \sup\limits_{\gamma_i\in\Lambda} \frac{1}{i}, & \text{if } \sup\limits_{\gamma_i\in\Lambda} \frac{1}{i} < 0.5; \\ 1 - \sup\limits_{\gamma_i\in\Lambda^c} \frac{1}{i}, & \text{if } \sup\limits_{\gamma_i\in\Lambda^c} \frac{1}{i} < 0.5; \\ 0.5, & \text{otherwise} \end{cases}$$

and the complex uncertain variables be defined by

$$\xi_i(\gamma_j) = \begin{cases} i, & \text{if } j = i; \\ 0, & \text{otherwise} \end{cases}$$

for $i \in I_r$ and $\xi \equiv 0$. For $\varepsilon > 0$, we have
$\lim\limits_{r\to\infty} M(\{\gamma \in \Gamma : \frac{1}{h_r}\sum\limits_{k\in I_r} \|\xi_k(\gamma) - \xi(\gamma)\| > \varepsilon\})$
$= \lim\limits_{r\to\infty} M(\{\gamma \in \Gamma : \frac{1}{h_r}\sum\limits_{k\in I_r} \|\xi_k(\gamma)\| > \varepsilon\})$
$= \lim\limits_{r\to\infty} M(\{\gamma_i\})$
$= \lim\limits_{r\to\infty} \frac{1}{i} \to 0 \qquad (\text{as} \quad i \in I_r)$

The sequence $\{\xi_i\}$ lacunary strongly converges in measure to ξ.

However for each $i \in I_r$, we have the uncertainty distribution of uncertain variable $\|\xi_i - \xi\| = \|\xi_i\|$ is

$$\Phi_i(x) = \begin{cases} 0, & \text{if } x < 0; \\ 1 - \frac{1}{i}, & \text{if } \ 0 \leq x < i; \\ 1, & \text{otherwise.} \end{cases}$$

$E[\frac{1}{h_r} \sum_{k \in I_r} \|\xi_k(\gamma) - \xi(\gamma)\|]$
$= \int_0^{+\infty} M\{\xi \geq x\}dx - \int_{-\infty}^0 M\{\xi \leq x\}dx$
$= \int_0^i 1 - (1 - \frac{1}{i})dx$
$= 1.$

That is, the $\{\xi_i\}$ does not converge in mean to ξ.

Lemma 4.1. *Assume complex uncertain sequence $\{\xi_n\}$ with real part $\{\zeta_n\}$ and imaginary part $\{\eta_n\}$, respectively, for $n = 1, 2, ...$ If uncertain sequences $\{\zeta_n\}$ and $\{\eta_n\}$ converge in measure to ζ and η, respectively, then complex uncertain sequence $\{\xi_n\}$ converge in measure to $\xi = \zeta + i\eta$.*

Theorem 4.2. *Assume complex uncertain sequence $\{\xi_k\}$ with real part $\{\zeta_k\}$ and imaginary $\{\eta_k\}$, respectively, for $n = 1, 2, ...$ If uncertain sequences $\{\zeta_k\}$ and $\{\eta_k\}$ lacunary strongly convergent in measure to ξ and η, respectively, then complex uncertain sequence $\{\xi_k\}$ lacunary strongly uniformly convergent in distribution to $\xi = \zeta + i\eta$.*

Proof. Let $c = a + ib$ be a point at which the complex uncertainty distribution Φ is continuous. For any $\alpha > a, \beta > b$, we have,

$$\begin{aligned} \{\zeta_k \leq a, \eta_k \leq b\} = \{&\zeta_k \leq a, \eta_k \leq b, \zeta \leq \alpha, \eta \leq \beta\} \\ &\cup \{\zeta_k \leq a, \eta_k \leq b, \zeta > \alpha, \eta > \beta\} \\ &\cup \{\zeta_k \leq a, \eta_k \leq b, \zeta \leq \alpha, \eta > \beta\} \\ &\cup \{\zeta_k \leq a, \eta_k \leq b, \zeta > \alpha, \eta \leq \beta\} \end{aligned}$$

$$\subset \{\zeta \leq \alpha, \eta \leq \beta\} \cup \{\|\zeta_k - \zeta\| \geq \alpha - a\} \cup \{\|\eta_k - \eta\| \geq \beta - b\}.$$

It follows from the subadditivity axiom that

$$\begin{aligned} \Phi_k(c) = \Phi_k(a + ib) \leq \Phi(\alpha + i\beta) &+ M\{\|\zeta_k - \zeta\| \geq \alpha - a\} \\ &+ M\{\|\eta_k - \eta\| \geq \beta - b\}. \end{aligned}$$

Since $\{\zeta_k\}$ and $\{\eta_k\}$ lacunary strongly convergent in measure to ζ and η respectively.

So for $\varepsilon > 0$ and $k \in I_r$ we have,

$$\lim_{r\to\infty} M\left\{\frac{1}{h_r}\sum_{k\in I_r}\|\zeta_k-\zeta\| \geq (\alpha-a)\geq \varepsilon\right\}=0$$

$$\text{and} \quad \lim_{r\to\infty} M\{\frac{1}{h_r}\sum_{k\in I_r}\|\eta_k-\eta\| \geq (\beta-b)\geq \varepsilon\}=0.$$

Thus we have, $\limsup_{r\to\infty} \ \Phi_k(c) \leq \Phi(\alpha+i\beta)$ for any $\alpha > a, \beta > b$.

Letting $\alpha + i\beta \to a + ib$, we get,

$$\limsup_{r\to\infty} \Phi_k(c) \leq \Phi(c). \tag{4.1}$$

On the other hand, for any $x < a, y < b$ we have,
$\{\zeta \leq x, \eta \leq y\} = \{\zeta_k \leq a, \eta_k \leq b, \zeta \leq x, \eta \leq y\}$
$\cup\{\zeta_k \leq a, \eta_k \leq b, \zeta \leq x, \eta \leq y\}$
$\cup\{\zeta_k > a, \eta_k \leq b, \zeta \leq x, \eta \leq y\} \cup \{\zeta_k > a, \eta_k > b, \zeta \leq x, \eta \leq y\}$

$$\subset \{\zeta_k \leq a, \eta_k \leq b\} \cup \{\|\zeta_k - \zeta\| \geq a - x\} \cup \{\|\eta_k - \eta\| \geq b - y\},$$

which implies

$$\Phi(x+iy) \leq \Phi_k(a+ib) + M\{\|\zeta_k-\zeta\| \geq a-x\} + M\{\|\eta_k-\eta\| \geq b-y\}.$$

Since

$$\lim_{r\to\infty} M\left\{\frac{1}{h_r}\sum_{k\in I_r}(\|\zeta_k-\zeta\| \geq a-x)\geq \varepsilon\right\}=0$$

$$\text{and} \quad \lim_{r\to\infty} M\left\{\frac{1}{h_r}\sum_{k\in I_r}(\|\eta_k-\eta\| \geq b-y)\geq \varepsilon\right\}=0,$$

$$\text{we obtain} \quad \Phi(x+iy) \leq \liminf_{r\to\infty} \Phi_k(a+ib)$$

for any $x < a, y < b$.
Taking $x + iy \to a + ib$, we get

$$\Phi(c) \leq \liminf_{r\to\infty} \Phi_k(c). \tag{4.2}$$

It follows from (4.1) and (4.2) that $\Phi_k(c) \to \Phi(c)$ as $r \to \infty$ and $k \in I_r$. That is the complex uncertain sequence $\{\xi_k\}$ is lacunary strongly convergent in distribution to $\xi = \zeta + i\eta$. □ □

Converse of the above theorem need not be true in general i.e. lacunary strongly convergent in distribution does not imply lacunary strongly convergence in measure. Following example illustrates this.
Consider the uncertainty space (Γ, L, M) to be $\{\gamma_1, \gamma_2\}$ with $M\{\gamma_1\} = M\{\gamma_2\} = \frac{1}{2}$. We define a complex uncertain variable as

$$\xi(\gamma) = \begin{cases} 1, & \text{if } \gamma = \gamma_1; \\ -1, & \text{if } \gamma = \gamma_2. \end{cases}$$

We also define $\{\xi_k\} = -\xi$, for $k \in I_r$. Then $\{\xi_k\}$ and ξ have the same distribution and thus $\{\xi_k\}$ converges in distribution to ξ. However, for any given $\varepsilon > 0$, we have

$$\lim_{r\to\infty} M \left\{ \gamma \in \Gamma : \tfrac{1}{h_r} \sum_{k\in I_r} \|\xi_k(\gamma) - \xi(\gamma)\| > \varepsilon \right\}$$
$$= \lim_{r\to\infty} M \left\{ \gamma \in \Gamma : \{ \tfrac{1}{h_r} \sum_{k\in I_r} \|2\xi_k(\gamma)\| > \varepsilon \right\}$$
$$\neq 0.$$

Therefore, the sequence $\{\xi_k\}$ does not lacunary strongly convergent in measure to ξ.

Theorem 4.3. *Let $\xi_1, \xi_2, \xi_3, ...$ be complex uncertain variables. Then $\{\xi_k\}$ is lacunary strongly convergent almost surely to ξ if and only if for any $\varepsilon > 0$, we have,*

$$M\left(\bigcap_{r\in I_{r_k}} \bigcup_{k\in I_r} \left\{ \gamma \in \Gamma : \frac{1}{h_r} \sum_{k\in I_r} \|\xi_k(\gamma) - \xi(\gamma)\| > \varepsilon \right\} \right) = 0.$$

Proof. By the definition of lacunary strongly convergence almost surely, we have that there exists an event Λ with $M(\Lambda) = 1$, such that

$$\lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} \|\xi_k(\gamma) - \xi(\gamma)\| = 0,$$

for every $\gamma \in \Lambda$.
Then for any $\varepsilon > 0$ there exists m such that $\frac{1}{h_r} \sum_{k\in I_r} \|\xi_k(\gamma) - \xi(\gamma)\| < \varepsilon$ where $k > m$, for any $\gamma \in \Lambda$, which is equivalent to

$$M\left(\bigcup_{r\in I_{r_k}} \bigcap_{k\in I_r} \left\{ \gamma \in \Gamma : \frac{1}{h_r} \sum_{k\in I_r} \|\xi_k(\gamma) - \xi(\gamma)\| > \varepsilon \right\} \right) = 1.$$

But using the duality axiom it follows that

$$M\left(\bigcap_{r\in I_{r_k}}\bigcup_{k\in I_r}\left\{\gamma\in\Gamma:\frac{1}{h_r}\sum_{k\in I_r}\|\xi_k(\gamma)-\xi(\gamma)\|>\varepsilon\right\}\right)=0.$$

Hence the result is proved. □

Theorem 4.4. *Let $\xi_1, \xi_2, \xi_3, ...$ be complex uncertain variables. Then $\{\xi_k\}$ is lacunary strongly convergent uniformly almost surely to ξ if and only if for any $\varepsilon > 0$, we have,*

$$\lim_{r\to\infty} M\left(\bigcup_{k\in I_r}\left\{\gamma\in\Gamma:\frac{1}{h_r}\sum_{k\in I_r}\|\xi_k(\gamma)-\xi(\gamma)\|>\varepsilon\right\}\right)=0.$$

Proof. If $\{\xi_k\}$ is lacunary strongly convergent uniformly almost surely to ξ then for any $\delta > 0$, there exists an event E such that $M\{E\} < \delta$ and $\{\xi_k\}$ converges uniformly to ξ on $\Gamma - E$. Thus for any $\varepsilon > 0$, there exists $m > 0$ such that $\frac{1}{h_r}\sum_{k\in I_r}\|\xi_k(\gamma)-\xi(\gamma)\| < \varepsilon$ where $k \geq m$ and $\gamma \in \Gamma - E$.

That is,

$$\bigcup_{k\in I_r}\left\{\gamma\in\Gamma:\frac{1}{h_r}\sum_{k\in I_r}\|\xi_k(\gamma)-\xi(\gamma)\|>\varepsilon\right\}\subset E$$

$$M\left(\bigcup_{k\in I_r}\{\gamma\in\Gamma:\frac{1}{h_r}\sum_{k\in I_r}\|\xi_k(\gamma)-\xi(\gamma)\|>\varepsilon\}\right)\leq M(E)<\delta$$

$$\text{Thus, } M\left(\bigcup_{k\in I_r}\{\gamma\in\Gamma:\frac{1}{h_r}\sum_{k\in I_r}\|\xi_k(\gamma)-\xi(\gamma)\|>\varepsilon\}\right)=0.$$

Conversely if,

$$M\left(\bigcup_{k\in I_r}\{\gamma\in\Gamma:\frac{1}{h_r}\sum_{k\in I_r}\|\xi_k(\gamma)-\xi(\gamma)\|>\varepsilon\}\right)=0$$

for any $\varepsilon > 0$, then for any given $\delta > 0$ and $k \geq 1$, there exists m_k such that

$$M\left(\bigcup_{k\in I_r}\{\gamma\in\Gamma:\frac{1}{h_r}\sum_{k\in I_r}\|\xi_k(\gamma)-\xi(\gamma)\|\geq\frac{1}{k}\}\right)<\frac{\delta}{2^k}.$$

Let,

$$E = \bigcup_{k \in I_r} \bigcup_{m_k \in I_{r_k}} \left\{ \gamma \in \Gamma : \frac{1}{h_r} \sum_{k \in I_r} \|\xi_k(\gamma) - \xi(\gamma)\| \geq \frac{1}{k} \right\}.$$

Then $M(E) < \delta$.
Also we have,

$$\sup_{\Gamma - E} \frac{1}{h_r} \sum_{k \in I_r} \|\xi_k(\gamma) - \xi(\gamma)\| \geq \frac{1}{k};$$

for any $k \in \mathbb{N}$ and $k > m_k$.
Hence the result. □

Theorem 4.5. *Let $\xi_1, \xi_2, \xi_3, ...$ be complex uncertain variables. If $\{\xi_k\}$ is lacunary strongly convergent uniformly almost surely to ξ, then $\{\xi_k\}$ is lacunary strongly convergent in measure to ξ.*

Proof. If $\{\xi_k\}$ is lacunary strongly convergent uniformly almost surely to ξ then

$$\lim_{r \to \infty} M \left(\bigcup_{k \in I_r} \{\gamma \in \Gamma : \frac{1}{h_r} \sum_{k \in I_r} \|\xi_k(\gamma) - \xi(\gamma)\| > \varepsilon\} \right) = 0$$

from the above theorem.
But, $M \left(\{\gamma \in \Gamma : \frac{1}{h_r} \sum_{k \in I_r} \|\xi_k(\gamma) - \xi(\gamma)\| > \varepsilon\} \right)$

$$\leq M \left(\bigcup_{k \in I_r} \{\gamma \in \Gamma : \frac{1}{h_r} \sum_{k \in I_r} \|\xi_k(\gamma) - \xi(\gamma)\| > \varepsilon\} \right).$$

Therefore, $\{\xi_k\}$ is lacunary strongly convergent in measure to ξ. □

5. Conclusion

This paper introduces different types of lacunary and lacunary strongly convergent as well as strongly Cesàro sequences of complex uncertain variables. Establish relations among the introduced notions. The work done can be further improved and applied for investigations from different aspects.

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